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SOME PROBLEMS OF THE UNIQUENESS OF THE RE-ESTABLISHMENT OF THE BRUNT–VÄISÄLÄ FREQUENCY USING DISPERSION CURVES†

V. V. RYNDINA

Rostov-on-Don

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The uniqueness of the solution of an equation for the square of the Brunt–Väisälä frequency (BVF), constructed using a sequence of dispersion curves for internal gravitational waves in an ocean of constant depth with a continuously changing BVF, is investigated in specific classes. Examples of functional classes in which the BVF is uniquely re-established using a sequence of dispersion curves are presented. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

A continuously stratified ocean of constant depth H is considered. According to a previously described [1] mathematical model of such an ocean, the dispersion curves of the internal gravitational waves are determined by the eigenvalues $\omega^2 = \omega_n^2(k^2)$ of the boundary-value problem

$$W'' - \frac{\mu(z)}{g}W' + \frac{\mu(z)-\omega^2}{\omega^2 - f^2}k^2 W = 0$$
$$z \in [-H, 0]; \quad W(-H) = 0, \quad W'(0) = g\frac{k^2}{\omega^2 - f^2}W(0) \tag{1.1}$$

where $\mu(z)$ is the square of the Brunt–Väisälä frequency (BVF), g is the acceleration due to gravity, f is the Coriolis parameter, k is the wave number and ω is the frequency of the free harmonic waves.

The determination of the stratification of an ocean which is quantitatively characterized by the BVF is one of the most important problems in oceanography. The problem of constructing the BVF using a known sequence of dispersion curves $\{\omega^2 = \omega_n^2(k^2)\}$, that is, the inverse eigenvalue problem for problem (1.1), is of great interest. This problem includes the questions of the uniqueness of the re-establishment of the BVF from the dispersion curves and the construction of algorithms for the re-establishment of the BVF from the dispersion curves in one or another class. Some of these questions have been considered previously [2–5]‡ (we correct a misprint: in formulae (2.2) of paper [2], there should be $\rho_*(x)$ instead of $\rho_*(z)$ under the integral signs).

Below, problems of the uniqueness of the re-establishment of the BVF using dispersion curves are investigated on the basis of the so-called fundamental equation for the BVF, which is determined by a known sequence of dispersion curves.

2. THE FUNDAMENTAL EQUATION

We will change from boundary-value problem (1.1), after changing to the dimensionless quantities

$$x = -\frac{z}{H}, \quad u = \frac{W}{(gH)^{1/2}}, \quad \eta^2 = \frac{H}{g}\omega^2, \quad \xi^2 = H^2k^2, \quad q(x) = \frac{H}{g}\mu(-Hx), \quad F^2 = \frac{H}{g}f^2$$

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‡See also: RYNDINA V. V., On the uniqueness of the re-establishment of the Brunt–Väisälä frequency from a sequence of dispersion curves, Rostov-on-Don, 1986; deposited in VINITI 17.03.86, No. 1806–B86; On the possibility of the re-establishment of a partially known Brunt–Väisälä frequency of a stratified ocean using known dispersion curves, Rostov-on-Don, 1989; deposited in VINITI, 31.03.89, No. 2571–B89; An algorithm for the re-establishment of a partially known Brunt–Väisälä frequency of a stratified ocean using partially known dispersion curves, Rostov-on-Don, 1990, deposited in VINITI, 23.01.90, No. 476–B90.

to the boundary-value problem

$$u'' + q(x)u' + \frac{q(x) - \eta^2}{\eta^2 - F^2}\xi^2 u = 0$$
$$u'(0) = -\xi^2(\eta^2 - F^2)^{-1}u(0), \quad u(1) = 0 \tag{2.1}$$

with the two parameters ξ and η, which represent the wave number and the frequency of the free harmonic waves respectively.

As in [1], we introduce the parameter $\lambda = \xi^2(\eta^2 - F^2)^{-1}$ into the treatment and, in addition, the parameter $s = \lambda^2 - \lambda F^2 - \xi^2$. At the same time, boundary-value problem (2.1) is rewritten in the form

$$(\rho u')' + (\lambda q - \lambda^2)\rho u = -s\rho u$$
$$u'(0) = -\lambda u(0), \quad u(1) = 0, \quad \rho(x) = \exp\left(\int_0^x q(t)dt\right) \tag{2.2}$$

Boundary-value problem (2.2) is equivalent to the integral equation with a symmetric kernel

$$y(x) = s\int_0^1 K(x,t,\lambda)y(t)dt, \quad y(x) = \sqrt{\rho(x)}u(x) \tag{2.3}$$

$$K(x,t,\lambda) = \exp[-\lambda(x+t)]\sqrt{\rho(x)\rho(t)}\int_\sigma^1 \frac{\exp(2\lambda y)}{\rho(y)}dy, \qquad \sigma = \begin{cases} t, & x \leq t \\ x, & x > t \end{cases} \tag{2.4}$$

For any real λ, the kernel $K(x, t, \lambda)$ satisfies the conditions in Finkel'shtein's theorem [6] for oscillatory behaviour and, consequently, it is an oscillatory kernel. Integral equation (2.3) therefore has a denumerable set of positive and simple characteristic values $s_0(\lambda) < s_1(\lambda) <$.

In order to explain how the functions $s_j(\lambda)$ are defined in terms of the functions $\eta_j^2(\xi^2) = g^{-1}H\omega_j^2(H^{-2}\xi^2)$,
which are assumed to be known in the inverse eigenvalue problem, we note that the pair of equations

$$\lambda = \xi^2(\eta^2 - F^2)^{-1}, \quad s_j(\lambda) = \lambda^2 - \lambda F^2 - \xi^2 \tag{2.5}$$

determine the dispersion curves $\eta^2 = \eta_j^2(\xi^2)$ of problem (2.1) in parametric form, subject to the condition that the parameter λ runs through a set of values of the function $\lambda_j(\xi^2) = \xi^2(\eta_j^2(\xi^2) - F^2)^{-1}$, observed on the ray $(0, +\infty)$. Since $\lambda_j(\xi^2)$ becomes larger in $(0, +\infty)$ [1], the set under consideration is the ray $(c_j, +\infty)$, where $c_j = \lim\lambda_j(\xi^2)$ when $\xi \to 0$.

It has been shown in [1] that $\eta_j^2(\xi^2) = F^2$ when $\xi \to 0$. Hence, putting $\eta_j^2(0) = F^2$, we obtain that c_j is the reciprocal of the angular coefficient of the tangent to the graph of the dispersion curves $\eta^2 = \eta_j^2(\xi^2)$ at the origin of the coordinate system in the plane (ξ^2, η^2).

It also follows from (2.5) that the function $s_j(\lambda)$ can be directly calculated using the formula

$$s_j(\lambda_j(\xi^2)) = \lambda_j^2(\xi^2) - F^2\lambda_j(\xi^2) - \xi^2$$

at the points $\lambda = \lambda_j(\xi^2)$ running through the ray $(c_j, +\infty)$ when ξ runs through the ray $(0, +\infty)$.

The function $s_j(\lambda)$ is holomorphic in a certain neighbourhood $G_j(I)$ of any segment I of the real line. This follows from the fact that the family of integral operators $K(\lambda)$ with a symmetric real kernel $K(x, t, \lambda)$ is a self-adjoint holomorphic family of type (A) of compact operators, defined on the real axis and that, for any $K(\lambda)$, the operator $\lambda \in R$ has simple characteristic values which do not vanish [7].

As an analytic function, $s_j(\lambda)$ with its values in the ray $(c_j, +\infty)$, is defined in the whole of its domain of analyticity and, in particular, $s_j(\lambda)$ is defined on the real line. Hence, the sequence of characteristic numbers $\{\omega^2 = \omega_j^2(k^2)\}$ of integral equation (2.3), which depends analytically on the non-eigenvalue parameter λ, is uniquely defined by the known sequence of dispersion curves $\{s_j(\lambda)\}$ of problem (1.1).

The trace of kernel (2.4)

$$A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{s_n(\lambda)} \tag{2.6}$$

can be represented in the form

$$A(\lambda)=\int_0^1 K(x,x,\lambda)dx=\int_0^1 \Phi(x)\exp(2\lambda x)dx,\quad \Phi(x)=\int_x^1 \frac{\rho(t-x)}{\rho(t)}dt \tag{2.7}$$

The function $\Phi(x)$ is defined in terms of $A(\lambda)$ using an inverse Laplace transformation and, consequently, the function $\Phi(x)$ is determined by the sequence of dispersion curves since, according to formula (2.6), the trace $A(\lambda)$ is determined by this sequence. The last equality of (2.7) can be looked upon as an equation which the function $\rho(x)$ satisfies.

Note that it follows from the definition of $\rho(x)$ in terms of $q(x)$ that the problems of finding $\rho(x)$ and $q(x)$ are equivalent.

The last equation of (2.7) is equivalent to the equation

$$\Phi(x)=\int_x^1 \exp\left[\int_t^{t-x} q(\zeta)d\zeta\right]dt \tag{2.8}$$

in $q(x)$. Equation (2.8) is the fundamental equation in this paper.

It can be shown by direct verification that, if the function $q(x)$ is a solution of Eq. (2.8), the function $q_1(x)=q(1-x)$ will also be its solution. It follows from this that Eq. (2.8) cannot, generally speaking, be solved uniquely in the class of continuous functions in the interval [0, 1]. There is therefore interest in the problem of picking out the functional classes Ψ, in which it is uniquely solvable, subject to the condition that the function $\Phi(x)$ is chosen from the class

$$G=\left\{\Phi(x)=\int_x^1 \exp\left[\int_t^{t-x} q(\zeta)d\zeta\right]dt : q(x)\in\Psi\right\}$$

We shall refer to such classes Ψ as classes of uniqueness for Eq. (2.8).

3. SOLUTION OF THE FUNDAMENTAL EQUATION IN THE CASE OF A PARTIALLY KNOWN BVF

We will consider the case when the function $q(x)\in C[0,1]$ is known in the interval $[a,1]$, $a\in(0,1)$. We note that, in a real ocean, the BVF changes appreciably at a comparatively shallow depth and is practically constant at considerable depths (more than 1 km). The case being considered is therefore of practical interest. We will show that, if the function $q(x)$ is equal to a known function $q_0(x)$ in the interval $[a,1]$, then it is uniquely determined from Eq. (2.8) in the interval $[0,1-a]$.

Replacing x with $1-x$ in the last equality of (2.7) and differentiating the resulting equality, we obtain

$$-\Phi'(1-x)=\frac{\rho(x)}{\rho(1)}+\int_0^x \frac{q(1-x+t)}{\rho(1-x+t)}\rho(t)dt \tag{3.1}$$

It follows from (3.1) that $\rho(1)=-(\Phi'(1))^{-1}$ is determined by the dispersion curves. If $0<t<x<1-a$, then $1-x+t\in[a,1]$ and, consequently, the quantities

$$q(1-x+t),\quad \rho(1-x+t)=\rho(1)\exp\left[\int_1^{1-x+t} q_0(\zeta)d\zeta\right]$$

are known for $t\in[0,x]$. The Volterra equation of the second kind (3.1) for $\rho(x)$ has a unique solution in the class of continuous functions in the interval $[0,1-a]$.

If $q(x) = q_0 = \text{const}$ in $[a, 1]$, then, for $x \in [0, 1-a]$, the equality

$$\exp(-q_0 x)\Phi(1-x) = -\Phi'(1)\int_0^x \rho(t)\exp(-q_0 t)dt \tag{3.2}$$

follows from the last relation of (2.7).

By differentiating this equation, we obtain the explicit representation

$$\rho(x) = -P[\Phi'(1-x) + q_0\Phi(1-x)], \quad P = \rho(1), \quad x \in [0, 1-a] \tag{3.3}$$

The case when $a = 1/2$ has been considered previously [4, 5] (see also the last two papers cited in the footnote). Note that the factor $K(x, t, \lambda)$ has been omitted in the formula of the kernel $\sqrt{\rho(x)\rho(t)}$ in [4, 5].

If it is known that $q(x) \in C_1[0, 1]$, then, when $a = 1/2$, there is no need to know that value q_0 beforehand since it can be found using the function $\Phi(x)$.

We will now show how this can be done. Differentiating relation (3.3) and putting $x = 1/2$ in the resulting equality and in equality (3.3), we obtain (everywhere henceforth up to the end of Section 3

$$\begin{gathered} \rho = \rho(1/2), \quad \Phi = \Phi(1/2), \quad \Phi' = \Phi'(1/2), \quad \Phi'' = \Phi''(1/2)) \\ q_0\rho = -P(-\Phi'' - q_0\Phi'), \quad \rho = -P(\Phi' + q_0\Phi) \end{gathered} \tag{3.4}$$

It follows from (3.4) that

$$\Phi'' = -2q_0\Phi' - q_0^2\Phi$$

The correctness of the equality

$$\Phi'^2 - \Phi\Phi'' = (\Phi' + q_0\Phi)^2 \tag{3.5}$$

follows from this.

Since $P > 0$, it follows from (3.4) that the expression in the brackets on the right-hand side of (3.5) is negative. Hence,

$$q_0 = (-\Phi' - B)/\Phi, \quad B = (\Phi'^2 - \Phi\Phi'')^{1/2}$$

4. SOME GENERAL NECESSARY CONDITIONS FOR THE SOLUTIONS OF THE FUNDAMENTAL EQUATION

We assume that the function $q(x)$ is $m - 1$ times differentiable in the interval $[0, 1]$. Then, the functions

$$\rho(x), \quad \rho_1(x) = \exp\left[\int_0^x q(1-t)dt\right]$$

will be m times continuously differentiable in $[0, 1]$. The obvious relation

$$\rho(x)\rho_1(1-x) = P$$

holds between the functions $\rho(x)$ and $\rho_1(x)$, and, 125c using this, we can rewrite Eq. (2.8) in the form

$$\int_x^1 \rho(t-x)\rho_1(1-t)dt = P\Phi(x) \tag{4.1}$$

Differentiating the identity (4.1) k times ($k = 1, 2, \dots, m$) and putting $x = 0$ and $x = 1$ in the resulting equality, we obtain, in explicit form, the conditions which the solutions of the fundamental equation satisfy

$$(-1)^k P\Phi_{(0)}^{(k)} = \sum_{j=0}^{k-1} \rho_{(0)}^{(j)}\rho_{1(1)}^{(k-j-1)} + \int_0^1 \rho_{(t)}^{(k)}\rho_1(1-t)dt$$
$$(-1)^k P\Phi_{(1)}^{(k)} = \sum_{j=0}^{k-1} \rho_{(0)}^{(j)}\rho_{1(0)}^{(k-j-1)}; \quad k = 1,2,\ldots,m \tag{4.2}$$

It is now possible to change from equalities (4.2) to equalities which explicitly relate the values of the function $q(x)$ and its derivatives at the points $x = 0$ and $x = 1$. For instance, when $m = 3$, after some reduction we obtain from equalities (4.2)

$$q(0)+q(1)=\Phi_1, \quad Q_1=\Phi_2, \quad Q_2=\Phi_3$$
$$q'(0)-q'(1)-q(0)q(1)=\Phi_4, \quad Q_3-4q(0)q(1)=\Phi_5 \tag{4.3}$$

where

$$\Phi_1 = P\Phi''(1), \quad \Phi_2 = -1-\Phi'(0), \quad \Phi_3 = \Phi''(0)-P\Phi''(1)$$
$$\Phi_4 = -P\Phi^{(3)}(1)-\Phi_1^2, \quad \Phi_5 = -\Phi^{(3)}(0)+\Phi_4-\frac{3}{2}\Phi_1^2, \quad Q_n = \int_0^1 q^n(t)dt$$

5. EXAMPLES OF PARAMETRIC CLASSES OF UNIQUENESS.

Using conditions (4.3), which the solutions of Eq. (2.8) must satisfy, certain parametric classes of uniqueness for Eq. (2.8) can be constructed.

As an example of such a class, we consider the class of functions

$$\Psi_1 = \{c\exp[-\alpha(z-a)^2]: c>0, \ \alpha>0, \ z\in[0,1]\}$$

in which a is an arbitrary fixed real number. The second and third equalities from (4.3) give two equations which c and α must satisfy

$$cp(\alpha)=\Phi_2, \quad c^2p(2\alpha)=\Phi_3; \quad p(\alpha)=\int_0^1 \exp[-\alpha(z-a)^2]dz \tag{5.1}$$

On dividing, term by term, the square of the first of equalities (5.1) by the second, we obtain an equation for α

$$f(\alpha)=\Phi_2^2\Phi_3^{-1}, \quad f(\alpha)=(p(\alpha))^2(p(2\alpha))^{-1} \tag{5.2}$$

It can be shown that the function $f(\alpha)$ decreases in $[0, +\infty]$, and it follows from this that Eq. (5.2) has a unique solution. The positive constant c is uniquely defined by any of the equations of system (5.1). Hence, the class Ψ_1 is a class of uniqueness for Eq. (2.8).

As a second example, we consider the class of functions

$$\Psi_2 = \{az^n+b: \min(a,a+b)>0, \ n>1, \ a\neq 0\}$$

The first, second and fourth equalities from (4.3) give three equations

$$a+2b=\Phi_1, \quad a+(n+1)b=(n+1)\Phi_2, \quad na+b(a+b)=-\Phi_4$$

The quantities a and b are uniquely found from the first two equations and n is uniquely found from the third equation. Hence, Ψ_2 is a class of uniqueness in the case of Eq. (2.8).

We will investigate the class

$$\Psi_3 = \{az+b: a+b>0, \ b>0\}$$

Note that, together with the function $q(z)$, it also contains the function $q(1-z)$ and therefore will not be a class of uniqueness for Eq. (2.8). We take the subclasses of the class Ψ_3 which do not simultaneously contain the functions $q(z)$ and $q(1-z)$

$$\Psi_{31} = \{az+b: a\geqslant 0, b>0\}, \quad \Psi_{32} = \{az+b: a\leqslant 0, a+b>0\}$$

From the first and fourth equalities of (4.3), we obtain the system of equations for a and b

$$a + 2b = \Phi_1, \quad b(a + b) = -\Phi_4$$

This system of equations has two solutions

$$a_1 = D, \; b_1 = \frac{1}{2}(\Phi_1 - D); \quad a_2 = -D, \quad b_2 = \frac{1}{2}(\Phi_1 + D)$$

$$(D = \sqrt{\Phi_1^2 + 4\Phi_4})$$

the first of which belongs to the subclass Ψ_{31} and the second belongs to the subclass Ψ_{32}. Consequently, the subclasses Ψ_{31} and Ψ_{32} are classes of uniqueness in the case of Eq. (2.8).

In a similar manner, it can be verified that the classes of functions

$$\Psi_4 = \{a(z-\lambda)^{-1} : a > 0, \lambda < 0\}, \quad \Psi_5 = \{a(z-\lambda)^{-1} : a > 0, \lambda > 1\}$$

$$\Psi_6 = \{a\exp(\lambda z) : a > 0, \lambda > 0\}, \quad \Psi_7 = \{a\exp(\lambda z) : a > 0, \lambda < 0\}$$

are classes of uniqueness in the case of Eq. (2.8).

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